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Ideals, Filters, and Supports in Pseudoeffect Algebras

Wu Jing[1]

Ideals, filters, local ideals, local filters, and supports in pseudoeffect algebras are defined and studied.

KEY WORDS: ideal; filter; support; pseudoeffect algebra.

1. INTRODUCTION AND PRELIMINARIES

In the beginning of 1990s, Kôpka and Chovanec (1994) have presented a new axiomatic model, difference posets. First this idea was applied to fuzzy set ideas in quantum mechanics and then presented in general algebraic form. Difference posets generalize quantum logic, orthoalgebras, as well as the set of all effects (i.e. the system of all Hermitian operators A on a Hilbert space H with $O \leq A \leq I$), which are important for modeling unsharp measurement in a Hilbert space quantum mechanics. A big advantage of difference poset is the possibility of handling self-orthogonal events. In 1994, Pulmannová, Foulis (Foulis and Bennett, 1994) and other authors studied a structure, now called an effect algebra, with a primary operation $\oplus$ slightly generalizing orthoalgebras. Here $a \oplus b$ is defined only for mutually excluded events a and b. But, as stressed, difference posets and effect algebras are practically the same thing, because $\oplus$ can be uniquely derived from $\ominus$ to be an effect algebra and vice versa.

In 2001, Dvurečenskij and Vetterlein (2001) introduced a structure of pseudo-effect algebras which generalize the effect algebras by dropping the commutativity. Unfortunately, not much is known about this structure.

In general, logicians often prefer to deal with filters which can be presumed to represent the modality of necessity or truth, algebraists generally prefer to think in terms of ideal which often figure prominently in representation theory (see Chovanec and Rybarikova (1998), Foulis, Greechie, and Ruttimann (1992)). The aim of this paper is to consider the ideals and filters in pseudoeffect algebras. In

[1] Department of Mathematics, University of Central Florida, Orlando, Florida, 32816-1364; e-mail: wu511944@pegasus.cc.ucf.edu.

the last section supports, local filters, and local ideals in pseudoeffect algebras are introduced and studied. These would help us to investigate the structure theory of pseudoeffect algebra much better.

A structure $(\mathcal{E}, +, 0, 1)$, where "+" is a partial binary operation and 0 and 1 are constants, is called a *pseudoeffect algebra* if, for all $a, b, c \in \mathcal{E}$, the following hold:

(E1) $a + b$ and $(a + b) + c$ exist if and only if $b + c$ and $a + (b + c)$ exist, and in this case, $(a + b) + c = a + (b + c)$;
(E2) There is exactly one $d \in \mathcal{E}$ and exactly one $e \in \mathcal{E}$ such that $a+d = e+a = 1$;
(E3) If $a + b$ exists, there are elements $d, e \in \mathcal{E}$ such that $a + b = d + a = b + e$;
(E4) If $1 + a$ or $a + 1$ exists, then $a = 0$.

If the hypotheses of (E1) are satisfied, we write $a + b + c$ for the element $(a + b) + c = a + (b + c)$.

In view of (E2), we may define the two unary operations $^\sim$ and $^-$ by requiring for any $a \in \mathcal{E}$

$$a + a^\sim = a^- + a = 1.$$

The following statements have been proved in Dvurečenskij and Vetterlein (2001).

Proposition 1.1. *Let $(\mathcal{E}, +, 0, 1)$ be a pseudoeffect algebra. For all $a, b, c \in \mathcal{E}$ we have the following:*

(1) $a + 0 = 0 + a = a$ *(i.e. 0 is a neutral element)*;
(2) $a + b = 0$ *implies* $a = b = 0$ *(positivity)*;
(3) $0^\sim = 0^- = 1$, $1^\sim = 1^- = 0$;
(4) $a^{\sim -} = a^{-\sim} = a$;
(5) $a + b = a + c$ *implies* $b = c$, *and* $b + a = c + a$ *implies* $b = c$ *(concellation laws)*;
(6) $a + b = c$ *iff* $a = (b + c^\sim)^-$ *iff* $b = (c^- + a)^\sim$.

Definition 1.2 Let $(\mathcal{E}, +, 0, 1)$ be a pseudoeffect algebra. We define for $a, b \in \mathcal{E}$

$$a \leq b \quad \text{iff} \quad a + c = b \quad \text{for some} \quad c \in \mathcal{E}.$$

Note that, from (E3), it is clear that

$$a \leq b \quad \text{iff} \quad d + a = b \quad \text{for some} \quad d \in \mathcal{E}.$$

Proposition 1.3. (Dvurečenskij and Vetterlein, 2001). *Let $(\mathcal{E}, +, 0, 1)$ be a pseudoeffect algebra. The following hold in $\mathcal{E}$ for all $a, b, c, d \in \mathcal{E}$:*

(1) *$\leq$ is a partial order on $\mathcal{E}$;*
(2) *$a \leq b$ iff $b^- \leq a^-$ iff $b^\sim \leq a^\sim$. That is, $^\sim$ and $^-$ are isomorphisms of the order of $\mathcal{E}$ onto the dual order of $\mathcal{E}$*;
(3) *If $a + b$ exists, $c \leq a$, and $d \leq b$, then $c + d$ exists*;
(4) *$a + b$ exists iff $a \leq b^-$ iff $b \leq a^\sim$;*
(5) *Suppose $b + c$ exists. Then $a \leq b$ if and only if $a + c$ exists and $a + c \leq b + c$. Suppose $c + b$ exists. Then $a \leq b$ if and only if $c + a$ exists and $c + a \leq c + b$.*

2. IDEALS AND FILTERS IN PSEUDOEFFECT ALGEBRAS

In this section we study ideals and filters in pseudoeffect algebras. In what follows $\mathcal{E}$ will be a pseudoeffect algebra.

We begin with

Definition 2.1 A nonempty subset $I \subseteq \mathcal{E}$ is called an idea (in $\mathcal{E}$) if

(I1) $a \in I, b \in \mathcal{E}, b \leq a$ implies $b \in I$;
(I2) $a \in I, b \in \mathcal{E}, a \leq b,$ and either $(b^- + a)^\sim \in I$ or $(a + b^\sim)^- \in I$ implies $b \in I$.

Definition 2.2 A nonempty subset $F \subseteq \mathcal{E}$ is called a filter (in $\mathcal{E}$) if

(F1) $a \in F, b \in \mathcal{E}, a \leq b$ implies $b \in F$;
(F2) $a \in F, b \in \mathcal{E}, b \leq a$, and either $a^- + b \in F$ or $b + a^\sim \in F$ implies $b \in F$.

An ideal I (resp. a filter F) is *proper* if $1 \notin I$ (resp. $0 \notin F$).

It is obvious that $0 \in I$ for any ideal I and $1 \in F$ for any filter F in $\mathcal{E}$.

Theorem 2.3 *A nonempty subset $I \subseteq \mathcal{E}$ is an ideal if and only if*
(I1) *$x \in I, y \in \mathcal{E}, y \leq x$ implies $y \in I$;*
(I2′) *$x \in I, y \in I, x \leq y^\sim$ implies $y + x \in I$.*

Proof: It suffices to show the equivalence between (I2) and (I2′).

(I2) $\Rightarrow$ (I2′). Suppose that $x \in I, y \in I,$ and $x \leq y^\sim$. By Proposition 1.3, $y + x$ exists. Now let $a = x, b = y + x$, then $a \leq b$ and $(a + b^\sim)^- = [x + (y + x)^\sim]^- = (y^\sim)^- = y \in I$, hence, by (I2), we have $b = y + x \in I$.

(I2′) $\Rightarrow$ (I2). If $a \in I, b \in \mathcal{E}, a \leq b,$ and $(b^- + a)^\sim \in I$, let $x = a, y = (b^- + a)^\sim$, then $x \in I, y \in I,$ and $y^- = b^- + a = b^- + x$, hence $x \leq y^-$, therefore $y \leq x^\sim$. Thus we have $x + y \in I$. But $x + y = a + (b^- + a)^\sim = b$, and so $b \in I$.

For the case of $a \in I, b \in I$, and $(a + b^\sim)^- \in I$, one can let $x = a$ and $y = (a + b^\sim)^-$. Now the rest of the proof goes similarly. □

Theorem 2.4 *A nonempty subset $F \subseteq \mathcal{E}$ is a filter if and only if*
(F1) $x \in F, y \in \mathcal{E}, x \leq y$ *implies* $y \in F$;
(F2′) $x \in F, y \in F, y^{\sim} \leq x$ *implies* $(y^{\sim} + x^{\sim})^{-} \in F$.

Proof: (F2) $\Rightarrow$ (F2′). If $x \in F$, $y \in F$, and $y^{\sim} \leq x$, let $a = x$, $b = (y^{\sim} + x^{\sim})^{-}$. From $(y^{\sim} + x^{\sim})^{-} + y^{\sim} + x^{\sim} = 1$, we have $(y^{\sim} + x^{\sim})^{-} + y^{\sim} = x$. Hence $(y^{\sim} + x^{\sim})^{-} \leq x$, that is, $b \leq a$. Since $a^{-} + b = [(y^{\sim} + x^{\sim})^{-} + y^{\sim}]^{-} + (y^{\sim} + x^{\sim})^{-} = y$, so $a^{-} + b \in F$, which leads to $(y^{\sim} + x^{\sim})^{-} = b \in F$.

(F2′) $\Rightarrow$ (F2). Now suppose that $a \in F$, $b \in \mathcal{E}$, $b \leq a$, and $a^{-} + b \in F$. Let $x = a$, $y = a^{-} + b$, then $x \in F$, $y \in F$. It is evident that $a^{-} + b + (a^{-} + b)^{\sim} = 1$, which implies that $b + (a^{-} + b)^{\sim} = a$, hence $y^{\sim} = (a^{-} + b)^{\sim} \leq a = x$. Therefore $(y^{\sim} + x^{\sim})^{-} \in F$. Since $(y^{\sim} + x^{\sim})^{-} = [(a^{-} + b)^{\sim} + (b + (a^{-} + b)^{\sim})^{\sim}]^{-} = (b^{\sim})^{-} = b$, and so $b \in F$.

For the case of $x \in F$, $b \in \mathcal{E}$, $b \leq a$, and $b + a^{\sim} \in F$, with the similar argument above one can complete the proof. □

Theorem 2.5 *If $I \subseteq \mathcal{E}$ is a proper ideal and $a \in I$, then neither a^{-} nor $a^{\sim}$ is in I.*

Proof: Assume on the contrary that $a^{-} \in I$. Let $x = a$, $y = a^{-}$, then $x = a = y^{\sim}$. By (I2′), we have $y + x = a^{-} + a = 1 \in I$, a contradiction. □

Similarly we have

Theorem 2.6 *If $F \subseteq \mathcal{E}$ is a proper filter and $a \in F$, then neither a^{-} nor $a^{\sim}$ is in F.*

Theorem 2.7 *Let I be a proper ideal of a pseudoeffect algebra $\mathcal{E}$, then both $I^{-} = \{a^{-} : a \in I\}$ and $I^{\sim} = \{a^{\sim} : a \in I\}$ are proper filters.*

Proof: (1) If a $\in I^{-}$, $b \in \mathcal{E}$, and $a \leq b$, then there is $x \in I$ such that $x^{-} = a$, hence $x^{-} \leq b$, and equivalently $b^{\sim} \leq x$, which yields that $b^{\sim} \in I$, therefore $b = (b^{\sim})^{-} \in I^{-}$.

Suppose that $a \in I^{-}$, $b \in \mathcal{E}$, $b \leq a$, and $a^{-} + b \in I^{-}$. Then there exist x and y in I such that $x^{-} = a$ and $y^{-} = a^{-} + b$. Then $y = (a^{-} + b)^{\sim}$, and $a = b + (a^{-} + b)^{\sim}$ since $a^{-} + b + (a^{-} + b)^{\sim} = 1$. Therefore $y \leq a = x^{-}$, that is, $x \leq y^{\sim}$. By (I2′), we obtain $y + x \in I$. But $y + x = (a^{-} + b)^{\sim} + [b + (a^{-} + b)^{\sim}]^{\sim} = b^{\sim}$ and so $b^{\sim} \in I$, furthermore, $b = (b^{\sim})^{-} \in I^{-}$.

If $a \in I^{-}$, $b \leq \mathcal{E}$, $b \leq a$, and $b + a^{\sim} \in I^{-}$, we can find x and y in I such that $x^{-} = a$ and $y^{-} = b + a^{\sim}$. Then $x = a^{\sim}$ and $y^{-} = b + x$, hence $x \leq y^{-}$, i.e., $y \leq x^{\sim}$. By (I2′), we get $x + y \in I$. Sinc $x + y = a^{\sim} + (b + a^{\sim})^{\sim} = b^{\sim}$, we have $b = (b^{\sim})^{-} \in I^{-}$.

(2) We now prove that $I^{\sim}$ is a proper filter.

If $a \in I^{\sim}$, $b \leq \mathcal{E}$, and $a \leq b$, then there is $x \in I$ such that $a = x^{\sim}$. We have $b^{-} \leq x$ as $a \leq b$. Since I is an ideal, we have $b^{-} \in I$, moreover, $b = (b^{-})^{\sim} \in I^{\sim}$.

Suppose that $a \in I^{\sim}, b \in \mathcal{E}, b \leq a$, and $a^{-} + b \in I^{\sim}$. Then we can find $x, y \in I$ such that $x^{\sim} = a$ and $y^{\sim} = a^{-} + b$. Then $y^{\sim} = x + b$, which implies that $x \leq y^{\sim}$. Applying (I2′) we have $y + x \in I$. Therefore $b^{-} = (a^{-} + b)^{-} + a^{-} = y + x \in I$, and so $b = (b^{-})^{\sim} \in I^{\sim}$.

If $a \in I^{\sim}, b \in \mathcal{E}, b \leq a$, and $b + a^{\sim} \in I^{\sim}$. Then there exist x and y in I such that $x^{\sim} = a$ and $y^{\sim} = b + a^{\sim}$. From $(b + a^{\sim})^{-} + b + a^{\sim} = 1$, we have $a = (b + a^{\sim})^{-} + b$, hence $y = (b + a^{\sim})^{-} \leq a = x^{\sim}$. By (I2′), we have $x + y \in I$. As $x + y = [(b + a^{\sim})^{-} + b]^{-} + (b + a^{\sim})^{-} = b^{-}$, then $b = (b^{-})^{\sim} \in I^{\sim}$, this completes the proof. □

Theorem 2.8 *Let F be a proper filter of a pseudoeffect algebra $\mathcal{E}$, then both $F^{\sim} = \{a^{\sim} : a \in F\}$ and $F^{-} = \{a^{-} : a \in F\}$ are proper ideals.*

Proof: If $x \in F^{\sim}$, $y \in \mathcal{E}$, and $y \leq x$, then $x = a^{\sim}$ for some $a \in F$. And so $y \leq a^{\sim}$, that is, $a \leq y^{-}$, hence $y^{-} \in F$, therefore $y = (y^{-})^{\sim} \in F^{\sim}$.

Now suppose that $x \in F^{\sim}$, $y \in F^{\sim}$, and $x \leq y^{\sim}$. We can find $a, b \in F$ such that $x = a^{\sim}$ and $y = b^{\sim}$. We have $y \leq x^{-}$ since $x \leq y^{\sim}$. And so $b^{\sim} \leq (a^{\sim})^{-} = a$. By (F2′), we have $(b^{\sim} + a^{\sim})^{-} \in F$, i.e. $(y + x)^{-} \in F$, and so $y + x \in F^{\sim}$. We have shown that $F^{\sim}$ is a proper ideal.

To complete the proof it remains to prove that F^{-} is a proper ideal too.

If $x \in F^{-}$, $y \in \mathcal{E}$, and $y \leq x$. Then $x = a^{-}$ for some $a \in F$, and so $y \leq a^{-}$, this implies that $a \leq y^{\sim}$, hence $y^{\sim} \in F$ and $y = (y^{\sim})^{-} \in F^{-}$.

If $x \in F^{-}$, $y \in F^{-}$, and $x \leq y^{\sim}$. Then we can choose $a, b \in F$ such that $x = a^{-}$, $y = b^{-}$. From $x \leq y^{\sim}$, we have $a^{-} \leq b$, and equivalently $b^{\sim} \leq a$. By Proposition 1.3, $b^{-} + a^{-}$ and $b^{\sim} + a^{\sim}$ exist. Since $a^{-} + (b^{-} + a^{-})^{\sim} = b$ and $(b^{-} + a^{-})^{\sim} \leq a$, by (F2), we get $(b^{-} + a^{-})^{\sim} \in F$, hence $y + x = b^{-} + a^{-} = [(b^{-} + a^{-})^{\sim}]^{-} \in F^{-}$. □

Corollary 2.9. *Let I be a proper ideal and F be a proper filter in a D-poset $\mathcal{P}$. Then $I^{\perp} = \{a^{\perp} : a \in I\}$ is a proper filter and $F^{\perp} = \{a^{\perp} : a \in F\}$ is a proper ideal in $\mathcal{P}$.*

Corollary 2.10. *Let I and F be subsets of a pseudoeffect algebra $\mathcal{E}$, then*

(1) *I^{-} is a proper ideal if and only if $I^{\sim}$ is a proper ideal*;

(2) *F^{-} is a proper filter if and only if $F^{\sim}$ is a proper filter.*

Definition 2.11 Let $\mathcal{E}$ and $\mathcal{F}$ be two pseudoeffect algebras. A mapping $\phi : \mathcal{E} \to \mathcal{F}$ is said to be a pseudoeffect morphism if

(1) $\phi(1) = 1$;

(2) if $a + b$ exists in $\mathcal{E}$, then $\phi(a) + \phi(b)$ exists in $\mathcal{F}$ and $\phi(a + b) = \phi(a) + \phi(b)$.

If, in addition, ϕ is bijective and ϕ^{-1} is also a pseudoeffect morphism, then ϕ is called a pseudoeffect isomorphism.

It is easy to verify the following proposition.

Proposition 2.12. *Let $\mathcal{E}$ and $\mathcal{F}$ be two pseudoeffect algebras and $\phi : \mathcal{E} \to \mathcal{F}$ be a pseudoeffect morphism, then*

(1) $\phi(0) = 0$;

(2) $\phi(a^-) = \phi(a)^-$ *and* $\phi(a^\sim) = \phi(a)^\sim$ *for every* $a \in \mathcal{E}$;

(3) $a \leq b$ *implies* $\phi(a) \leq \phi(b)$.

Theorem 2.13. *Let $\mathcal{E}$ and $\mathcal{F}$ be two pseudoeffect algebras and $\phi : \mathcal{E} \to \mathcal{F}$ be a pseudoeffect morphism. Let $\mathcal{K} = \{a \in \mathcal{E} : \phi(a) = 0\}$ and $\mathcal{Y} = \{a \in \mathcal{E} : \phi(a) = 1\}$, then $\mathcal{K}$ is a proper ideal and $\mathcal{Y}$ is a proper filter in $\mathcal{E}$.*

Proof: If $x \in \mathcal{K}$, $y \in \mathcal{E}$, and $y \leq x$. Then $\phi(y) \leq \phi(x)$. Since $\phi(x) = 0$, so $\phi(y) = 0$, therefore $y \in \mathcal{K}$.

Suppose that $x \in \mathcal{K}$, $y \in \mathcal{K}$, and $x \leq y^\sim$. Then $y + x$ exists and $\phi(y + x) = \phi(y) + \phi(x) = 0$, hence $y + x \in \mathcal{K}$.

As $\phi(1) = 1$, so $1 \notin \mathcal{K}$, hence $\mathcal{K}$ is a proper ideal.

With the similar approach one can show that $\mathcal{Y}$ is a proper filter. □

Theorem 2.14. *Let $\mathcal{E}$ and $\mathcal{F}$ be two pseudoeffect algebras and $\phi : \mathcal{E} \to \mathcal{F}$ be a pseudoeffect isomorphism, then I is an ideal in $\mathcal{E}$ if and only if $\phi(I)$ is an ideal in $\mathcal{F}$. Moreover, I is a proper ideal in $\mathcal{E}$ if and only if $\phi(I)$ is a proper ideal in $\mathcal{F}$.*

Proof: Notice that ϕ preserves the partial order $\leq$ in both directions. Now the proof goes easily. □

Similarly we have

Theorem 2.15. *Let $\mathcal{E}$ and $\mathcal{F}$ be two pseudoeffect algebras and $\phi : \mathcal{E} \to \mathcal{F}$ be a pseudoeffect isomorphism, then F is a filter in $\mathcal{E}$ if and only if $\phi(F)$ is a filter in $\mathcal{F}$. Moreover, F is a proper filter in $\mathcal{E}$ if only if $\phi(F)$ is a proper filter in $\mathcal{F}$.*

3. SUPPORTS, LOCAL FILTERS, AND LOCAL IDEALS IN PSEUDOEFFECT ALGEBRAS

Throughout this section $\mathcal{E}$ will be a pseudoeffect algebras. We first introduce the definition of supports in pseudoeffect algebras as follows.

Definition 3.1 A subset S of $\mathcal{E}$ is called a support iff $0 \notin S$ and, for each pair $p, q \in \mathcal{E}$ with $p + q$ exists in $\mathcal{E}$,

$$p + q \in S \Leftrightarrow \{p, q\} \cap S \neq \emptyset.$$

Notice that the empty set $\emptyset$ is a support. A nonempty support is called a *proper* support. It is easy to see that S is proper if and only if $1 \in S$.

Theorem 3.2. *Let S be a support in $\mathcal{E}$. If $a \in S$, $b \in \mathcal{E}$, and $a \leq b$, then $b \in S$.*

Proof: We note that if $a \leq b$ then $b = a + (b^- + a)^\sim$. Now the proof goes easily. □

Definition 3.3 A triple $\{p, q, r\}$ in $\mathcal{E}$ is called a right triangle if $p + r$, $q + r$, $p + (q + r)$, and $q + (p + r)$ exist in $\mathcal{E}$, and is denoted by $\Delta(p, q \rhd r)$.

A triple $\{p, q, r\}$ is called a left triangle in $\mathcal{E}$ if $r + p$, $r + q$, $(r + p) + q$ and $(r + q) + p$ exist in $\mathcal{E}$, and we denote it by $\Delta(r \lhd p, q)$.

We now define the local filters in pseudoeffect algebras.

Definition 3.4 A nonempty subset F of $\mathcal{E}$ is called a loacal filter if for every right triangle $\Delta(p, q \rhd r)$ in $\mathcal{E}$

$$p + r, q + r \in F \Leftrightarrow r \in F.$$

Dually we can also define.

Definition 3.4 A nonempty subset F of $\mathcal{E}$ is called a local filter if for every left triangle $\Delta(r \lhd p, q)$ in $\mathcal{E}$

$$r + p, r + q \in F \Leftrightarrow r \in F.$$

Indeed, we have the following.

Theorem 3.5. *Definition* 3.4 *and* 3.4′ *are equivalent.*

Proof: We only prove Definition 3.4 ⇒ Definition 3.4′, and the converse is analogous. Let $\Delta(r \lhd p, q)$ be a left triangle in $\mathcal{E}$. We can choose $a, b \in \mathcal{E}$ such that $r + p = a + r$ and $r + q = b + r$. It is easy to check that $a = [r + (r + p)^\sim]^-$. We claim that triple $\{a, b, r\}$ is a right triangle. Indeed, by $(r + p) + q + [(r + p) + q]^\sim = 1$, we have $(r + p)^\sim = q + [(r + p) + q]^\sim$, hence $q \leq (r + p)^\sim$. From $r \leq r + p = [(r + p)^\sim]^-$, we can infer that $r + (r + p)^\sim$ exists in $\mathcal{E}$. By Proposition 1.3, we have $r + q \leq r + (r + p)^\sim$, and so $a = [r + (r + p)^\sim]^- \leq (r + q)^- = (b + r)^-$. Again by Proposition 1.3, we conclude that $a + (b + r)$ exists in $\mathcal{E}$. Similarly we can show that $b + (a + r)$ exists in $\mathcal{E}$, hence triple

$\{a, b, r\}$ is a right triangle. Now the rest of the proof is straightforward, so we omit it. □

In fact in the proof above we have proved the following result.

Theorem 3.6. *Let p, q, r, a, b be elements of $\mathcal{E}$. If $p + r = r + a$ and $q + r = r + b$, then $\{p, q, r\}$ is a right triangle if and only if $\{a, b, r\}$ is a left triangle.*

Theorem 3.7. *Every filter in pseudoeffect algebra is a local filter.*

Proof: Let F be a filter and $\Delta(p, q \rhd r)$ a right triangle in pseudoeffect algebra $\mathcal{E}$. If $r \in F$, then, obviously, $p + r, q + r \in F$. Suppose now that $p + r, q + r \in F$. Let $a = p + r$ and $b = r$, then $a \in F$ and $b \leq a$. From $[q + (p + r)]^{-} + q + (p + r) = 1$, we have $[q + (p + r)]^{-} + q = (p + r)^{-}$. Then $a^{-} + b = [q + (p + r)]^{-} + q + r \in F$ as $q + r \in F$ and F is a filter. By condition (F2) in Definition 2.2, we have $r = b \in F$. □

If S is a support in $\mathcal{E}$, we let

$$F_S^{(-)} = \{p \in \mathcal{E} : p^{-} \notin S\}.$$

and

$$F_S^{(\sim)} = \{p \in \mathcal{E} : p^{\sim} \notin S\}.$$

The following theorem show the relation between supports and local filters.

Theorem 3.8. *Let S be a support in $\mathcal{E}$, then both $F_S^{(-)}$ and $F_S^{(\sim)}$ are local filters.*

Proof: We now prove that $F_S^{(-)}$ is a local filter. Without loss of generality, we suppose that S is a proper support. Let $\Delta(p, q \rhd r)$ is a right triangle.

We claim that if $a \in F_s^{(-)}$, $b \in \mathcal{E}$, and $a \leq b$, then $b \in F_s^{(-)}$. In fact, by $a \leq b$, we have $b^{-} \leq a^{-}$. Since $a^{-} \notin S$, by Theorem 3.2, b^{-} is not in S, that is, $b \in F_s^{(-)}$. Then if $r \in F_s^{(-)}$, we have that $p + r, q + r \in F_s^{(-)}$.

Now let $p + r, q + r \in F_S^{(-)}$. By $[q + (p + r)]^{-} + q + (p + r) = 1$, we get $(p + r)^{-} = [q + (p + r)]^{-} + q$. As $p + r \in F_S^{(-)}$, that is, $(p + r)^{-} \notin S$, it follows that $[q + (p + r)]^{-} + q \notin S$. Then we can infer that $q \notin S$. By $q + r \in F_S^{(-)}$, we have $[p + (q + r)]^{-} + p = (q + r)^{-} \notin S$. Since $r^{-} = [(p + q + r)^{-} + p] + q$ and S is a support, it follow that $r^{-} \notin S$, i.e. $r \notin F_s^{(-)}$.

With the same argument one can show that $F_s^{(\sim)}$ is also a local filter. □

To introduce that definition of local ideals in pseudoeffect algebras we need the following result.

Theorem 3.9. *Let F be a nonempty subset of $\mathcal{E}$, then $F^{\sim}$ is a local filter if and only if F^{-} is a local filter.*

Proof: Suppose that $F^{\sim}$ is a local filter, and $\Delta(p, q \rhd r)$ is a right triangle in $\mathcal{E}$.

If $r \in F^{-}$, then $r^{\sim\sim} \in F^{\sim}$. Since $r^{\sim\sim} \leq (p+r)^{\sim\sim}, (q+r)^{\sim\sim}$, we have that $(p+r)^{\sim\sim}, (q+r)^{\sim\sim} \in F^{\sim}$ as $F^{\sim}$ is a local filter. Then $(p+r)^{\sim}, (q+r)^{\sim} \in F$, which yields that $p+r, q+r \in F^{-}$.

Now suppose that $p+r, q+r \in F^{-}$. Then $(p+r)^{\sim\sim}, (q+r)^{\sim\sim} \in F^{\sim}$. Since $r^{\sim\sim} \leq (p+r)^{\sim\sim}, (q+r)^{\sim\sim}$, we can find $a, b \in \mathcal{E}$ such that $a + r^{\sim\sim} = (p+r)^{\sim\sim}$ and $b + r^{\sim\sim} = (q+r)^{\sim\sim}$.

We now show that triple $\{a, b, r \sim\sim\}$ is a right triangle.

From $a + r^{\sim\sim} = (p+r)^{\sim\sim}$, we have $r^{\sim\sim} = [(p+r)^{\sim} + a]^{\sim}$. Then $(p+r)^{\sim} + a = r^{\sim}$, furthermore, $a = [r + (p+r)^{\sim}]^{\sim}$. Since $\Delta(p, q \rhd r)$ is a right triangle, so $p + (q+r)$ exists in $\mathcal{E}$, and so $q + r \leq p^{\sim}$. Obviously, $p^{\sim} = r + (p+r)^{\sim}$, hence $q + r \leq r + (p+r)^{\sim}$, equivalently, $[r + (p+r)^{\sim}]^{\sim} \leq (q+r)^{\sim}$. It follows that $a = [r + (p+r)^{\sim}]^{\sim} \leq (q+r)^{\sim} = [(q+r)^{\sim\sim}]^{-} = (b + r^{\sim\sim})^{-}$, hence $a + (b + r^{\sim\sim})$ exists in $\mathcal{E}$. Similarly, we can prove that $b + (a + r^{\sim\sim})$ exists in $\mathcal{E}$, and so $\{a, b, r^{\sim\sim}\}$ is a right triangle.

Since $F^{\sim}$ is a local filter, and $a + r^{\sim\sim}, b + r^{\sim\sim} \in F^{\sim}$, then $r^{\sim\sim} \in F^{\sim}$, it follows that $r \in F^{-}$, which completes the proof. □

We are now in a position to define.

Definition 3.10 A nonempty subset I of $\mathcal{E}$ is called a local ideal if I^{-} (equivalently, $I^{\sim}$) is a local filter.

By Theorem 2.7, it is obvious that every ideal in pseudoeffect algebra is a local ideal.

If S be a support in $\mathcal{E}$, we let $I_S = \mathcal{E} \backslash S = \{p \in \mathcal{E} : p \notin S\}$.

Theorem 3.11. *Let S be a support in $\mathcal{E}$, then I_S is a local ideal.*

Proof: Note that $I_S^{\sim} = F_S^{(-)}$. By Theorem 3.8, we have that I_S is a local ideal. □

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